From the standpoint of rigor, the treatment of the calculus by these men was far in advance of the Continental. In Great Britain there was achieved in the eighteenth century in the geometrical treatment of fluxions that which was not achieved in the algebraical treatment until the nineteenth century. It was not until after the time of Weierstrass that infinitesimals were cast aside by mathematical writers on the Continent.

Judged by modern standards all eighteenth century expositions of the calculus, even the best British expositions, are defective. As pointed out by Landen and Woodhouse, there was an unnaturalness in founding the calculus upon *motion* and *velocity*. These notions apply in a real way only to dynamics. Moreover, not all continuous curves can be conceived as traceable by the motion of a point. The notion of variable velocity is encumbered with difficulties. Then again, in all discussion of limits during the eighteenth century, the question of the *existence of a limit* of a given sequence was never raised. The word "quantity" was not defined; quantities were added, subtracted, multiplied and divided. Were these quantities numbers, or were they considered without reference to number? Both methods are possible. Which did British authors follow? No explicit answer to this was given. Our understanding of authors like Maclaurin, Rowe and others, is that in initial discussions such phrases as "fluxion of a curvilinear figure" are used in a non-arithmetical sense; the idea is purely geometrical. When later the finding of the fluxions of terms in the equations of curves is taken up, the arithmetical or algebraical conception is predominant. Rarely does a writer speak of the difference between the two. Perhaps

> "His notions fitted things so well
> That which was which he could not tell."

The theory of irrational number caused no great anxiety to eighteenth century workers. Operations applicable to rational numbers were extended without scruple to a domain of numbers which embraced both rational and irrational. There was no careful exposition of the number system used. The modern theories of irrational number have brought about the last stages of what is called the *arithmetization* of mathematics. As now developed in books which aim at rigor the notion of a limit makes no reference to quantity and is a purely ordinal notion. Of this mode of treatment the eighteenth century had never dreamed.

NOTE ON SOME APPLICATIONS OF A GEOMETRICAL TRANSFORMATION TO CERTAIN SYSTEMS OF SPHERES.[1]

By DR. HENRY W. STAGER, Fresno, California.

In a paper in Volume VI of the *Proceedings of the Edinburgh Mathematical Society*, Professor Allardice considers the transformation *in plano:*

[1] Presented to the San Francisco Section of the American Mathematical Society, April 12, 1913.

Let l be a fixed straight line, C any plane curve, t a tangent to C meeting l in X and making an angle θ with l; then C', the transformed curve of C, is the envelope of a straight line t' through X making an angle ϕ with l, where θ and ϕ are connected by the relation

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta.$$

The length of the tangent from any point in l to C remains unaltered by the transformation. A similar method is applicable in space and the following paper gives some of the results in its applications to certain systems of spheres. The method of transformation is as follows:

Let α be a fixed plane, S any surface, and β a plane tangent to S intersecting α in the line i and making with α an angle θ. If β' be a plane through i making with α an angle ϕ, determined by the relation

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta, \tag{1}$$

then the envelope of β' will be defined as the transformed surface of S, or, more simply, the "*transform of S.*"

From the given relation, $\tan \frac{1}{2}\phi = k \tan \frac{1}{2}\theta$, and the identity,

$$\tan \phi = \frac{2 \tan \phi/2}{1 - \tan^2 \phi/2},$$

we find that

$$\tan \phi = \frac{2k \tan \theta}{\pm (1 - k^2)\sqrt{1 + \tan^2 \theta} + 1 + k^2}. \tag{2}$$

It is evident from the nature of this relation that for every value of k there are two values of $\tan \phi$, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$.

For simplicity, we will use Cartesian coördinates to find the equation of the transform of a given sphere from its equation, and we will take the xy-plane as the plane of transformation, α. Let θ be the angle formed by the xy-plane and the plane $ux + vy + wz - 1 = 0$, and ϕ be the angle formed by the xy-plane and the plane $ux + vy + w_1z - 1 = 0$. (Since the intercepts of both planes on the x-axis and on the y-axis are equal, respectively $1/u$ and $1/v$, they intersect the xy-plane in the same line, as required by the transformation.)

From the formula for the angle between two planes,

$$\tan^2 \delta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(AA' + BB' + CC')^2},$$

we have

$$\tan^2 \theta = \frac{u^2 + v^2}{w^2} \qquad \text{and} \qquad \tan^2 \phi = \frac{u^2 + v^2}{w_1^2}.$$

Substituting these values in equation (2) and solving for w, we obtain

$$w = \frac{(1 + k^2)w_1 \pm (1 - k^2)\sqrt{u^2 + v^2 + w_1^2}}{2k}, \tag{3}$$

or,

$$(4)\quad w^2 = \frac{2(1+k^4)w_1^2 + (1-k^2)^2(u^2+v^2) \pm 2w_1(1-k^4)\sqrt{u^2+v^2+w_1^2}}{4k^2}.$$

Now, let the center of the sphere lie in the z-axis and its equation will be

$$(5)\qquad x^2 + y^2 + z^2 + 2nz + d = 0.$$

Expressing the condition that the plane $ux + vy + wz - 1 = 0$ be tangent to this sphere, we have

$$(6)\qquad (d - n^2)u^2 + (d - n^2)v^2 + dw^2 + 2nw + 1 = 0,$$

which is the tangential equation of the sphere.

Substituting the values of w and w^2 found in (3) and (4) and then simplifying and factoring the result, we have the equation of the transform of the given sphere

$$(7)\qquad \begin{aligned}&\left[\{(k^2+1)^2d - 2n^2(k^4+1) + 2n(k^4-1)\sqrt{n^2-d}\}(u^2+v^2) + 4dk^2w_1^2\right.\\ &\quad\left.+ 4k^2 + 4k\{n(k^2+1) + (k^2-1)\sqrt{n^2-d}\}w_1\right] \times \left[\{(k^2+1)^2d\right.\\ &\quad - 2n^2(k^4+1) - 2n(k^4-1)\sqrt{n^2-d}\}(u^2+v^2) + 4dk^2w_1^2 + 4k^2\\ &\quad\left.+ 4k\{n(k^2+1) - (k^2-1)\sqrt{n^2-d}\}w_1\right] = 0.\end{aligned}$$

This equation breaks up into two equations which are the tangential equations of two spheres, and may be written in the form

$$(8)\qquad A_1(u^2+v^2) + Cw_1^2 + 2N_1w_1 + D = 0;$$

$$(9)\qquad A_2(u^2+v^2) + Cw_1^2 + 2N_2w_1 + D = 0.$$

The Cartesian equations of these forms are

$$(10)\qquad (CD - N_1^2)x^2 + (CD - N_1^2)y^2 + A_1Dz^2 + A_1C + 2A_1N_1z = 0;$$

$$(11)\qquad (CD - N_2^2)x^2 + (CD - N_2^2)y^2 + A_2Dz^2 + A_2C + 2A_2N_2z = 0,$$

where, from the actual values of the coefficients involved,

$$(CD - N_1^2) = A_1D, \qquad \text{and} \qquad (CD - N_2^2) = A_2D.$$

Simplifying and substituting, we have finally

$$(12)\quad 4k^2(x^2+y^2+z^2) + 4k\{n(k^2+1) + (k^2-1)\sqrt{n^2-d}\}z + 4dk^2 = 0;$$

$$(13)\quad 4k^2(x^2+y^2+z^2) + 4k\{n(k^2+1) - (k^2-1)\sqrt{n^2-d}\}z + 4dk^2 = 0:$$

the equations of two spheres whose centers lie in the z-axis. Hence, in general, a sphere is transformed into two spheres whose centers lie in a line perpendicular to the plane of transformation and passing through the center of the given sphere.

Now, let ρ represent the radius of the given sphere and δ the distance of its

center from the plane of transformation; also, let r' and r'' and d' and d'' represent respectively the radii and distances from the plane of transformation of the centers of the transformed spheres. The equations of the latter may then be written

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) + (k - 1/k)\rho]z + d = 0; \tag{14}$$

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) - (k - 1/k)\rho]z + d = 0: \tag{15}$$

whence the resulting relations:

$$\begin{aligned} d' + r' &= (\delta - \rho)k; \qquad & d'' + r'' &= (\delta + \rho)k; \\ d' - r' &= (\delta + \rho)/k; \qquad & d'' - r'' &= (\delta - \rho)/k. \end{aligned} \tag{16}$$

We will next consider for what values of k the transformed spheres will degenerate into points. It is evident that we must consider each case separately.

In the first case we have

$$r' = \left| -\frac{1}{2k}\{(\delta + \rho) - k^2(\delta - \rho)\} \right| = 0,$$

whence

$$k^2 = (\delta + \rho)/(\delta - \rho). \tag{17}$$

Here we have two real values for k, provided the given sphere does not cut the plane of transformation; and a zero or infinite value for k if the given sphere is tangent to that plane. The corresponding values of the distance are

$$d' = \mp \sqrt{\delta^2 - \rho^2},$$

or in terms of the original coefficients,

$$d' = \mp \sqrt{d}. \tag{18}$$

In the second case, by similar methods, we find that

$$k^2 = (\delta - \rho)/(\delta + \rho), \tag{19}$$

and

$$d'' = \mp \sqrt{\delta^2 - \rho^2}, \qquad \text{or,} \qquad d'' = \mp \sqrt{d}. \tag{20}$$

It follows that, in general, a sphere may be transformed into a point for any one of four values of k, the points being the same in pairs, and all four being equidistant from the plane of transformation, one pair on each side. It is to be especially noted that while each of the two transformed spheres may degenerate into points, both spheres cannot become points at the same time, unless either $\rho = 0$, or $\delta = 0$; *i. e.*, when the given sphere is itself a point and this point may be transformed into its image with respect to the plane of transformation, or when the center of the given sphere lies in the plane of transformation and the points are imaginary. We may also note that a point can be transformed into only one sphere.

We have already seen, equations (12) and (13), that for the same value of k a given sphere may be transformed into two distinct spheres. These two spheres evidently result from the two values of ϕ, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$. In case the angle is θ we will say that the transformed sphere is traced out by a *direct* movement, and where the angle is $(\theta + \pi)$ we will say that it is traced out by an *indirect,* or *inverse,* movement. It is evident that if we transform two spheres, *both* directly, or *both* inversely, the *direct* common tangent planes to the two spheres become common tangent planes to the transformed spheres, while if one sphere is transformed directly and the other inversely, the *transverse* common tangent planes become common tangent planes to the transformed spheres.

Forming the equation of the radical plane of the given sphere and either of its transforms, we have

$$z = 0,$$

which is also the plane of transformation. It follows that the plane of transformation is the radical plane of the given sphere and its transforms, and that the length of a tangent line from any point in the plane of transformation to the sphere is unaltered by the transformation. Hence the distance between the points of contact of common tangent planes to two spheres remains unaltered. This is the important property of the transformation. By transforming several spheres into points at the same time, relations and properties of spheres may be obtained directly from known relations between points.

For two or more spheres to be transformed simultaneously into points, it is evidently sufficient that the values of k^2 be equal. Considering the case of two spheres with radii ρ' and ρ'' and with the distances of their centers from the plane of transformation, δ' and δ'', respectively, we have

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' - \rho'')/(\delta'' + \rho'');$$

whence

$$(\delta' - \rho')/(\delta' + \rho') = (\delta'' - \rho'')/(\delta'' + \rho''), \quad \text{or} \quad \delta'/\delta'' = \rho'/\rho''.$$

This is the condition that the plane of transformation pass through the direct center of similitude of the two spheres. Using the reciprocal values for k^2 we obtain the same result. Finally, let

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' + \rho'')/(\delta'' - \rho'');$$

whence

$$\delta'/\delta'' = - \rho'/\rho'':$$

the condition that the plane of transformation pass through the inverse center of similitude of the two spheres. We conclude that two spheres may be transformed simultaneously into points if the plane of transformation contain a center of similitude of the two spheres. Likewise, if the plane of transformation contain any one of the four axes of similitude of three spheres, these spheres will all be transformed into points by the same transformation. Further, if the plane

of transformation be any one of the eight planes of similitude of four spheres,[1] the four spheres may be transformed into points simultaneously; and, finally, any number of spheres which have a common plane of similitude may be so transformed. Two spheres on the same side of the plane of transformation will be transformed, either both directly, or both inversely; while of two on opposite sides, one will be transformed directly, the other inversely.

By giving k every possible value a given sphere may be transformed into an infinite series of spheres which, by virtue of the properties of the plane of transformation as the radical plane of a sphere and its transforms, is cut orthogonally by a system of spheres. Then, since the given sphere is one of the series, if we transform the whole system for the same value of k, it will transform into itself.

It may be noted at this point that the results here obtained by analytic methods may readily be obtained by synthetic methods, and in the sequel they will be interpreted in accordance with the particular problem under consideration. In all problems which follow it will be necessary to associate only those common tangent planes and distances between points of contact of common tangent planes to two spheres which determine the centers of similitude in the planes of similitude used as planes of transformation. Thus we will associate with the plane of similitude passing through the six direct centers of similitude of four spheres the external common tangent planes of each pair of spheres, and only the external common tangent planes of these spheres.

Some Applications of the Method.

1. The condition that four planes passing through a point should be tangent to the same sphere is given by the equation

$$\tan \tfrac{1}{2}\,\phi' \tan \tfrac{1}{2}\,\phi'' = \tan \tfrac{1}{2}\,\theta' \tan \tfrac{1}{2}\,\theta'',$$

where ϕ', ϕ'', θ', θ'' are the angles formed by the given planes and the plane determined by the lines of intersection of the two pairs of opposite planes.

Transform the sphere into a point with the external diagonal plane as the plane of transformation, and let the angles of the transformed planes with this plane of transformation be ϕ and θ, corresponding respectively to the angles ϕ', ϕ'' and θ', θ''. We then have the following relations:

$$\tan \tfrac{1}{2}\,\phi = k \tan \tfrac{1}{2}\,\phi'; \qquad \tan \tfrac{1}{2}\,\theta = k \tan \tfrac{1}{2}\,\theta';$$

$$\tan \tfrac{1}{2}\,(\phi + \pi) = k \tan \tfrac{1}{2}\,\phi''; \qquad \tan \tfrac{1}{2}\,(\theta + \pi) = k \tan \tfrac{1}{2}\,\theta''.$$

Eliminating ϕ, θ, k from these equations, we have the given condition.

[1] The twelve centers of similitude of four spheres lie in sets of six in a plane: the six direct centers lie in a plane; the three direct centers of any three spheres lie in a plane with the three inverse centers not paired with them; and any two direct centers, using each sphere only once, lie in a plane with the four inverse centers not paired with them. Thus there are eight planes of similitude, one containing all the direct centers; four containing three direct and three inverse centers; and three containing two direct and four inverse centers.

2. If five spheres have a common plane of similitude they may be transformed into five points by the same transformation. Therefore the distances between the points of contact of the common tangent planes of each pair of spheres satisfy the relation connecting the ten straight lines joining five points in space.

Let d_1, d_2, d_3, $\cdots$ d_{10} be the distances. Then the relation is given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 & 1 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 & 1 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 & 1 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 & 1 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

3. If, in addition to the conditions of No. 2, the five spheres also touch a sixth sphere, the ten distances will satisfy the additional relation connecting five points lying on the surface of a sphere.

Transforming the five spheres into points as above, they must lie on the surface of the transform of the sixth sphere and will therefore satisfy the additional relation given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 \end{vmatrix} = 0.$$

4. The radius R of a given sphere S tangent to four given spheres may be found in terms of R, the distance δ of the center of S from the plane containing the six direct centers of similitude of the four spheres, the distances between the points of contact of common tangent planes to pairs of spheres, and the radius and distance from its center to the plane of similitude of any one of the given spheres.

If we transform the four spheres into points by the same transformation, these four points will lie on the transform of S. Hence

$$R' = \frac{1}{2}\sqrt{\frac{(\Sigma aa')[\Pi(-aa' + bb' + cc')]}{\Sigma[a^2a'^2(b^2 + c^2 - a^2 + b'^2 + c'^2 - a'^2)] - (a^2b^2c^2 - \Sigma a^2b'^2c'^2)}},$$

where a, a', etc., are the distances between the points of contact of the common

[1] Cayley, "Collected Mathematical Papers," Volume I, p. 1 et seq.

tangent planes of two pairs of spheres with no sphere in common, and where Σ and π represent cyclo-symmetrical functions only. But

$$R' = \left| \frac{1}{2k} \{\delta(k^2 - 1) \mp (k^2 + 1)R\} \right|,$$

where

$$k^2 = (d_1 - r_1)/(d_1 + r_1), \qquad \text{or} \qquad k^2 = (d_1 + r_1)/(d_1 - r_1),$$

from equations (17), and (19).

Hence

$$R = | - \delta(r_1/d_1) \pm 2\{(\sqrt{d_1^2 - r_1^2})/d\}R' |.$$

5. The locus of the center of a sphere S such that the distances between the points of contact of the common tangent planes to S and four fixed spheres are equal is a straight line perpendicular to a plane of similitude of the four spheres.

Transforming the fixed spheres into points, the transformed spheres S' will form a system of spheres with a common center; namely, the center of the sphere determined by the four points. Moreover, all the spheres S' with real tangents to the four points lie within this sphere; consequently the distance between the points of contact cannot exceed the radius of this sphere, and hence it is a maximum when S' becomes a point; *i. e.*, when the four fixed spheres and S have a common plane of similitude. The theorem follows directly from the fact that the system S' is a system of concentric spheres. There are eight cases for consideration corresponding to the eight planes of similitude of the four fixed spheres.

6. The centers of the spheres tangent to four given spheres lie in pairs on eight straight lines passing through the radical center and perpendicular to one of the planes of similitude of the four spheres.

This is only a special case of the preceding. It is evident that the radical plane of any pair of the tangent spheres is the plane of similitude, which is perpendicular to the line joining the centers of the two spheres. The results of this theorem can be employed in the construction of the spheres tangent to four given spheres.

7. If S is a sphere tangent to four given spheres, which are tangent to one another, the five spheres have common planes of similitude.

Transforming the four spheres into points, the points will coincide, since each sphere touches one of the others. But the four points must lie on S', the transform of S, and therefore S' must also reduce to a point simultaneously with the other four; hence the result.

8. The locus of the centers of all spheres which touch a given sphere and have a common plane of similitude with it is an ellipsoid of revolution.

Let r be the radius and d be the distance from the common plane of similitude of the center of the given sphere, and let ρ and δ be the radius and distance respectively of the variable sphere. Transform the spheres into points for the same value of k and we have

$$d' = \frac{d}{2}(k + 1/k) - \frac{r}{2}(k - 1/k); \qquad \delta' = \frac{\delta}{2}(k + 1/k) - \frac{\rho}{2}(k - 1/k).$$

But, since all the spheres are tangent to the given sphere, the points into which they transform are coincident, and therefore, $d'' = \delta'$. Hence

$$(k - 1/k)(r + \rho) = \left(d + \delta - \frac{4d'}{k + 1/k}\right)(k + 1/k),$$

or, $(r + \rho)/(\delta + c) = e$, where e and c are constants.

Therefore, the locus of the centers of the spheres is the locus of a point, the ratio of whose distances from the center of the given sphere to its distance from a plane parallel to and at the distance c from the common plane of similitude is constant; *i. e.*, an ellipsoid of revolution.

THE ASTROLABE.

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Among ancient civilized peoples for many centuries the most important instrument used by astronomers, astrologers and surveyors was the astrolabe, or, more properly, the astrolabe planisphere.

The word is derived from two Greek words meaning "to follow the stars," and is therefore applicable in general to any astronomical or astrological instrument. Indeed, the name has been applied to at least three distinct forms. The first of these, better known as the armillary sphere, or *instrumentum armillarum* (instrument of rings), consists of two, three, or more brass circles, representing the circles of the celestial sphere, hinged together in the proper relations, and bearing tubes for sighting the heavenly bodies. An engraving of this may be found on the title page of many an old book on mathematics or astronomy. It was doubtless invented by Hipparchus, and was used by Ptolemy, who described it in the *Almagest*, Book V, Proposition 1. A large wooden ring, used by mariners in the days of Columbus and Vasco da Gama to take the altitude of the sun, was also known as an astrolabe. The most important form, however, was the astrolabe planisphere, or, simply, the planisphere.

The theory of the planisphere was given in the second century of our era by Ptolemy in a treatise on the planisphere, which is not extant in the Greek but survived in the Arabic, from which it was translated by Commandinus.

The Arabs, probably applying the theory of Ptolemy, constructed exceedingly accurate astrolabes. Remains of astrolabes have been found in Assyrian excavations, and the instrument is still of practical value in India and Persia. From Arabia it passed again into Europe, and was used by astronomers and surveyors in Italy, England and other countries until the eighteenth century. Bion wrote a treatise on the astrolabe in 1702, and Boileau mentioned it in a satire written in 1693:

> "Une astrolabe en main, elle a dans sa gouttière
> À suivre Jupiter passé la nuit entière."